ITERATION FORMULA FOR SOLVING THE EQUALITY CONSTRAINTS IN GRG

First consider the simple case with everything scalar

$$h(d, s) = 0 \qquad \text{(all scalars)} \qquad (1)$$

The first order perturbation is

$$\partial h = \frac{\partial h}{\partial d}\partial d + \frac{\partial h}{\partial s}\partial s \qquad (2)$$

For a known d, $\partial d = 0$ and $\partial h \neq 0$ unless we are at the solution. Therefore, from (2)

$$\partial h = \frac{\partial h}{\partial s}\partial s \quad \text{or} \quad h_{j+1} - h_j = \left(\frac{\partial h}{\partial s}\right)_j (s_{j+1} - s_j) \qquad (3)$$

In Eq. (3) we have $h_j \neq 0$ and <u>we want</u> $h_j = 0$, so we set

$$h_{j+1} = h_j + \left(\frac{\partial h}{\partial s}\right)_j (s_{j+1} - s_j) = 0 \qquad (4)$$

and solving for s_{j+1} we get

$$s_{j+1} = s_j - \left(\frac{\partial h}{\partial s}\right)_j^{-1} h_j \qquad (5)$$

Now we can generalize for vector $\underset{\sim}{h}$ at a point $\underset{\sim}{x}_{k+1} = (\underset{\sim}{d}_{k+1}, \underset{\sim}{s}_{k+1})^T$ of the <u>optimization</u> iteration k (the "outer iteration):

$$\left[\underset{\sim}{s}_{k+1}\right]_{j+1} = \left[\underset{\sim}{s}_{k+1} - \left(\frac{\partial \underset{\sim}{h}}{\partial \underset{\sim}{s}}\right)_{k+1}^{-1} \underset{\sim}{h}(\underset{\sim}{d}_{k+1}, \underset{\sim}{s}_{k+1})\right]_j \qquad (6)$$

where j is now the index of the "inner" iterations. This is Eq. (5.41) in the text.

The remaining question is how to initialize the iteration formula 6, i.e., what is the first guess for $\underset{\sim}{s}_{k+1}$ at $j=0$. A good guess is to pick the value that would satisfy the <u>linear</u> approximation of the constraints for a given change $\partial \underset{\sim}{d}$ in two consecutive outer iterations, i.e., $\partial \underset{\sim}{d} = \underset{\sim}{d}_{k+1} - \underset{\sim}{d}_k$, with the corresponding $\partial \underset{\sim}{s} = \underset{\sim}{s}'_{k+1} - \underset{\sim}{s}_k$ where $\underset{\sim}{s}'_{k+1}$ is the value that solves the <u>linear</u> approximation of $\underset{\sim}{h} = \underset{\sim}{0}$ (and $\underset{\sim}{s}_{k+1}$ being the value that solves the actual nonlinear $\underset{\sim}{h} = \underset{\sim}{0}$ at the conclusion of the j iterations).

To compute $\underset{\sim}{s}'_{k+1}$ we go to Eq. (2) and
↳ (vector form now)
set $\partial \underset{\sim}{h} = \underset{\sim}{0}$:

$$\partial \underset{\sim}{h} = \frac{\partial \underset{\sim}{h}}{\partial \underset{\sim}{d}} \partial \underset{\sim}{d} + \frac{\partial \underset{\sim}{h}}{\partial \underset{\sim}{s}} \partial \underset{\sim}{s} = \underset{\sim}{0}$$

Solve for $\partial \underset{\sim}{s}$:

$$\partial \underset{\sim}{s} = -\left(\frac{\partial \underset{\sim}{h}}{\partial \underset{\sim}{s}}\right)^{-1} \left(\frac{\partial \underset{\sim}{h}}{\partial \underset{\sim}{d}}\right) \partial \underset{\sim}{d} \qquad (7)$$

or with the iteration index,

$$\underset{\sim}{s}'_{k+1} - \underset{\sim}{s}_k = -\left(\frac{\partial \underset{\sim}{h}}{\partial \underset{\sim}{s}}\right)_k^{-1} \left(\frac{\partial \underset{\sim}{h}}{\partial \underset{\sim}{d}}\right)_k \left(\underset{\sim}{d}_{k+1} - \underset{\sim}{d}_k\right) \quad (8)$$

Finally, if we choose the gradient method to iterate in the reduced space $\underset{\sim}{d}$, in order to find the minimum, we have

$$\partial \underset{\sim}{d} = \underset{\sim}{d}_{k+1} - \underset{\sim}{d}_k = -\alpha_k \left(\frac{\partial z}{\partial \underset{\sim}{d}}\right)_k^T \quad (9)$$

where $\partial z / \partial \underset{\sim}{d}$ is the reduced gradient. Substituting (9) into (8) we get the formula that computes the initial guess $\underset{\sim}{s}'_{k+1}$ to be used for $j=0$ in iteration (6):

$$\underset{\sim}{s}'_{k+1} = \underset{\sim}{s}_k + \alpha_k \left(\frac{\partial \underset{\sim}{h}}{\partial \underset{\sim}{s}}\right)_k^{-1} \left(\frac{\partial \underset{\sim}{h}}{\partial \underset{\sim}{d}}\right)_k \left(\frac{\partial z}{\partial \underset{\sim}{d}}\right)_k^T .$$

This is Eq. (5.39) in the text.