ANALYTIC METHODS

Problem	Method	Step	Description	Condition
min $f(\mathbf{x})$	Unconstrained Optimality Conditions	1	Find all stationary points (points that satisfy $\nabla f = \mathbf{0}$)	$\nabla f(\mathbf{x}_\dagger) = \mathbf{0}$
		2	If the Hessian is positive definite at a stationary point, the point is a minimum	$\partial \mathbf{x}^T \left(\nabla^2 f\left(\mathbf{x}_\dagger\right)\right) \partial \mathbf{x} > 0$
min $f(\mathbf{x})$ s.t. $\mathbf{h}(\mathbf{x}) = \mathbf{0}$	Reduced Gradient	1	Split variables **x** into decision variables **d** and state variables **s** such that the number of state variables equals the number of independent constraints	$f(\mathbf{x}) \Rightarrow f(\mathbf{d},\mathbf{s})$
		2	Write perturbations of **h**(**d**,**s**) using a first order approximation. From a feasible point **x**, perturbations ∂**x** must be such that ∂**h = 0**	$\partial \mathbf{h} = \frac{\partial \mathbf{h}}{\partial \mathbf{s}} \partial \mathbf{s} + \frac{\partial \mathbf{h}}{\partial \mathbf{d}} \partial \mathbf{d} = \mathbf{0}$
		3	Solve for ∂**s** as a function of ∂**d** (this restricts movement in **x** to the constraint surface by restricting ∂**s** with respect to ∂**d**)	$\therefore \partial \mathbf{s} = -\left(\frac{\partial \mathbf{h}}{\partial \mathbf{s}}\right)^{-1} \left(\frac{\partial \mathbf{h}}{\partial \mathbf{d}}\right) \partial \mathbf{d}$
		4	Write the first order approximation of the perturbations for *f*(**d**,**s**), and substitute for ∂**s**	$\partial f = \frac{\partial f}{\partial \mathbf{d}} \partial \mathbf{d} + \frac{\partial f}{\partial \mathbf{s}} \partial \mathbf{s} = \left(\frac{\partial f}{\partial \mathbf{d}} - \frac{\partial f}{\partial \mathbf{s}} \left(\frac{\partial \mathbf{h}}{\partial \mathbf{s}}\right)^{-1} \frac{\partial \mathbf{h}}{\partial \mathbf{d}} \right) \partial \mathbf{d}$
min $f(\mathbf{x})$ s.t. $\mathbf{h}(\mathbf{x}) = \mathbf{0}$, $\mathbf{g}(\mathbf{x}) \le \mathbf{0}$	Monotonicity Analysis	1	Split $f(\mathbf{x})$ into monotonic regions	
		2	For each region, write the monotonicity table	
		3	Use MP1 and MP2 to determine constraint activity in order to eliminate a variable and/or constraints	**MP1**: Every decreasing variable is bounded below by at least one active constraint (and vise versa). **MP2**: Every non-objective variable is bounded above by at least one constraint and below by at least one constraint. If such a variable can be proven to be well constrained in both directions, then it can be eliminated by combining the bounding constraints.
		4	Rewrite the reduced problem and repeat #3 until solution, or if the problem is reduced to an unconstrained problem, use unconstrained optimality conditions to find the minimum	
	KKT Conditions	1	Write the Lagrangian function	$L(\mathbf{x}) = f(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{h} + \boldsymbol{\mu}^T \mathbf{g}$
		2	Solve system of equations with respect to **x**, **μ**, and **λ** to find KKT points $\mathbf{x}_\dagger$ • <u>Feasibility</u>: the constraints are satisfied • <u>Stationarity</u>: the gradient of the Lagrangian function is zero • <u>Transversality</u>: $\mu = 0$ for inactive constraints • <u>Positivity</u>: Lagrange multipliers μ are nonnegative and λ are nonzero	$\mathbf{h}\left(\mathbf{x}_\dagger\right) = \mathbf{0},\ \mathbf{g}\left(\mathbf{x}_\dagger\right) \le \mathbf{0}$ $\nabla L\left(\mathbf{x}_\dagger\right) = \mathbf{0}$ $\boldsymbol{\mu}^T \mathbf{g}\left(\mathbf{x}_\dagger\right) = \mathbf{0}$ $\boldsymbol{\lambda} \ne \mathbf{0},\ \boldsymbol{\mu} \ge \mathbf{0}$
		3	If the Hessian of the Lagrangian at the KKT point is positive definite in the subspace tangent to the active constraints, then the point is optimal	$\partial \mathbf{x}^T \left(\nabla^2 L\left(\mathbf{x}_\dagger\right)\right) \partial \mathbf{x} > 0 \quad \forall \partial \mathbf{x} : \nabla \mathbf{h} \partial \mathbf{x} = \mathbf{0}$

ITERATIVE METHODS

Problem	Method	Step	Description	Equation
min $f(\mathbf{x})$	Modified Steepest Descent	1	1st order approximation to f	$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) d\mathbf{x}$
		2	Line search in the $-\nabla f$ direction (find a good value for α using line search)	$\mathbf{x}_{n+1} = \mathbf{x}_n - \alpha \nabla f(\mathbf{x}_n)$
		3	Iterate to convergence	
	Modified Newton's Method	1	2nd order approximation to f	$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) d\mathbf{x} + \frac{1}{2} d\mathbf{x}^T \left(\nabla^2 f(\mathbf{x}_0)\right) d\mathbf{x}$
		2	Find minimum of the approximation using $\nabla f = \mathbf{0}$	$\nabla f(\mathbf{x}) \approx \nabla f(\mathbf{x}_0) + \nabla^2 f(\mathbf{x}_0) d\mathbf{x} = \mathbf{0}$ $\therefore d\mathbf{x} = -\left(\nabla^2 f\right)^{-1} \nabla f$
		3	Line search the actual function in the direction of the approximated minimum (find a good value for α using line search)	$\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha \left(\nabla^2 f_k\right)^{-1} \nabla f_k$
		4	Iterate to convergence	
**min $f(\mathbf{x})$ s.t. $\mathbf{h}(\mathbf{x}) = \mathbf{0}$	GRG Generalized Reduced Gradient	1	Split variables **x** into decision variables **d** and state variables **s** such that the number of state variables equals the number of independent constraints	$f(\mathbf{x}) \Rightarrow f(\mathbf{d}, \mathbf{s})$
		2	Write perturbations of **h**(**d**,**s**) using a first order approximation. From a feasible point **x**, perturbations $\partial\mathbf{x}$ must be such that $\partial\mathbf{h} = \mathbf{0}$	$\partial\mathbf{h} = \frac{\partial\mathbf{h}}{\partial\mathbf{s}}\partial\mathbf{s} + \frac{\partial\mathbf{h}}{\partial\mathbf{d}}\partial\mathbf{d} = \mathbf{0}$
		3	Solve for $\partial\mathbf{s}$ as a function of $\partial\mathbf{d}$ (this restricts movement in **x** to the constraint surface by restricting $\partial\mathbf{s}$ with respect to $\partial\mathbf{d}$)	$\therefore \partial\mathbf{s} = -\left(\frac{\partial\mathbf{h}}{\partial\mathbf{s}}\right)^{-1}\left(\frac{\partial\mathbf{h}}{\partial\mathbf{d}}\right)\partial\mathbf{d}$
		4	Write the first order approximation of the perturbations for $f(\mathbf{d},\mathbf{s})$, and substitute for $\partial\mathbf{s}$	$\partial f = \frac{\partial f}{\partial\mathbf{d}}\partial\mathbf{d} + \frac{\partial f}{\partial\mathbf{s}}\partial\mathbf{s} = \left(\frac{\partial f}{\partial\mathbf{d}} - \frac{\partial f}{\partial\mathbf{s}}\left(\frac{\partial\mathbf{h}}{\partial\mathbf{s}}\right)^{-1}\frac{\partial\mathbf{h}}{\partial\mathbf{d}}\right)\partial\mathbf{d}$
		5	The new *unconstrained* function z(**d**) is a projection of the $f(\mathbf{x})$ function onto the **d** space	$\therefore \frac{\partial z}{\partial\mathbf{d}} = \frac{\partial f}{\partial\mathbf{d}} - \frac{\partial f}{\partial\mathbf{s}}\left(\frac{\partial\mathbf{h}}{\partial\mathbf{s}}\right)^{-1}\left(\frac{\partial\mathbf{h}}{\partial\mathbf{d}}\right) = \mathbf{0}$
		6	Solve the unconstrained *z*(**d**) minimization problem using one step of the *Steepest Descent* algorithm to find $\mathbf{d}_{k+1}$ (*Newton's Method* could be used instead)	$\mathbf{d}_{k+1} = \mathbf{d}_k - \alpha_k \left(\frac{\partial\mathbf{z}}{\partial\mathbf{d}}\right)_k^T$
		7	Solve for $\mathbf{s}'_{k+1}$ using the linearized constraint found in step 4 and the $\mathbf{d}_{k+1}$ found in step 7. $\mathbf{s}'_{k+1}$ will not necessarily be on the true constraint surface	$\mathbf{s}'_{k+1} = \mathbf{s}_k - \left(\frac{\partial\mathbf{h}}{\partial\mathbf{s}}\right)_k^{-1}\left(\frac{\partial\mathbf{h}}{\partial\mathbf{d}}\right)_k\left(-\alpha_k\left(\frac{\partial\mathbf{z}}{\partial\mathbf{d}}\right)_k^T\right)$
		8	Solve for the true $\mathbf{s}_{k+1}$ using *Steepest Descent* (or *Newton's Method*) with the true constraint $\mathbf{h}(\mathbf{d}_{k+1},\mathbf{s}_{k+1}) = \mathbf{0}$ with starting point $\mathbf{s}'_{k+1}$ from step 8 (this is an inner iteration with index *j*). Here, the value of $\mathbf{h}(\mathbf{d}_{k+1},\mathbf{s}_{k+1})_j$ functions as the step length for the inner iteration. Iterate to convergence, but if convergence fails, goto step 7 and use a smaller α_k.	$\left[\mathbf{s}_{k+1}\right]_{j+1} = \left[\mathbf{s}_{k+1} - \mathbf{h}(\mathbf{d}_{k+1},\mathbf{s}_{k+1})\left(\frac{\partial\mathbf{h}}{\partial\mathbf{s}}\right)_{k+1}^{-1}\right]_j$
		9	Goto step 7 using the updated function for $\partial\mathbf{z}/\partial\mathbf{d}$ (as calculated in step 6) and iterate until **d** converges (solving the inner iteration for **s** at each iteration *k*)	$\left(\frac{\partial z}{\partial\mathbf{d}}\right)_{k+1} = \left(\frac{\partial f}{\partial\mathbf{d}}\right)_{k+1} - \left(\frac{\partial f}{\partial\mathbf{s}}\right)_{k+1}\left(\frac{\partial\mathbf{h}}{\partial\mathbf{s}}\right)_{k+1}^{-1}\left(\frac{\partial\mathbf{h}}{\partial\mathbf{d}}\right)_{k+1}$
	SQP Sequential Quadratic Programming	1	Generate the QP approximation. This is equivalent to using Newton's method to solve for $\nabla L(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{0}$ • 2nd order approximation to the Lagrangian • 1st order approximation of the constraints	$\underset{\mathbf{s}_k}{\text{minimize}}\ \hat{f}(\mathbf{s}_k) = L_k + (\nabla_{\mathbf{x}} L_k)\mathbf{s}_k + \frac{1}{2}\mathbf{s}_k^T\left(\nabla_{\mathbf{x}}^2 L_k\right)\mathbf{s}_k$ subject to $\hat{\mathbf{h}}(\mathbf{s}_k) = \mathbf{h}_k + \nabla\mathbf{h}_k\mathbf{s}_k = \mathbf{0}$
		2	Solve the QP sub-problem to find search direction $\mathbf{s}_k$. KKT conditions are linear, so the system of equations can be solved explicitly, or a more sophisticated method can be used.	$\nabla_{\mathbf{s}_k}\hat{L}(\mathbf{s}_k) = \nabla_{\mathbf{s}_k}\hat{f}(\mathbf{s}_k) + \hat{\boldsymbol{\lambda}}^T\nabla_{\mathbf{s}_k}\hat{\mathbf{h}}(\mathbf{s}_k) = \mathbf{0}$ $\therefore \nabla_{\mathbf{x}} L_k + \mathbf{s}_k^T\left(\nabla_{\mathbf{x}}^2 L_k\right) + \hat{\boldsymbol{\lambda}}^T\nabla\mathbf{h}_k = \mathbf{0}$
		3	Line search a penalty (merit) formulation of the original problem in the direction of $\mathbf{s}_k$ (find a good value for α using line search)	$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{s}_k$

** An active set strategy can be used for problems with inequality constraints